Some Undecidable Fragments of First-Order Modal Logic

Mo Liu

Department of Philosophy, PKU

June 5th, 2018

The Decision Problem

- "Entscheidungsproblem", Hilbert and Ackermann 1928.
  "The decision problem is solved when we know a procedure that allows, for any given logical expression, to decide by finitely many operations its validity or satisfiability. . . . The decision problem must be considered the main problem of mathematical logic."
- To devise a process (find an algorithm) that solves the satisfiability problem for first-order logic. ($Sat(FO)$).

Decistion Problem

- To devise a process (Algorithm): Effective Calculability, Computability
- Partial recursive function, Turing computable function, λ-calculus and so on...
- Negative answer to Entscheidungsproblem.
  - Church, 1936
  - Turing, 1937
- Church–Turing Thesis

The Decision Problem Afterwards

Which fragments of FO are decidable and which are undecidable?

The Decision Problem Afterwards

Which fragments of FO are decidable and which are undecidable?

- Classes defined by models.

The Decision Problem Afterwards

Which fragments of FO are decidable and which are undecidable?

- Classes defined by models.
- Prefix-Vocabulary Classes

$$[\Pi, (p_1, p_2, \cdots), (f_1, f_2, \cdots)]_{(=)}$$

The Decision Problem Afterwards

Which fragments of FO are decidable and which are undecidable?

- Classes defined by models.
- Prefix-Vocabulary Classes

$$[\Pi, (p_1, p_2, \cdots), (f_1, f_2, \cdots)]_{(=)}$$

- Fragments with only a bounded number of variables.

$$FO = \bigcup_{k=1}^{\infty} FO^k$$

Turing Computable Function

- Turing machine
- Turing computable function
- Undecidable problems, e.g. the Halting Problem.

Reduction

- A reduction is an algorithm for transforming problem **A** into problem **B**. This reduction may be used to show that **B** is at least as difficult as **A**.

Reduction

- A reduction is an algorithm for transforming problem **A** into problem **B**. This reduction may be used to show that **B** is at least as difficult as **A**.
- To show that a decision problem **P** is undecidable we must find a reduction from a decision problem which is already known to be undecidable. That reduction function must be a computable function.

Undecidability of FOL, Turing 1937

"Corresponding to each computing machine $\mathcal{M}$ we construct a formula $Un(\mathcal{M})$ and we show that, if there is a general method for determining whether $Un(\mathcal{M})$ is provable, then there is a general method for determining whether $\mathcal{M}$ ever prints 0."

Turing [1937], pp.259

Finite Model Property

In many cases, the decidability of the satisfiability problem for a formula class has been proved by showing that the given class Λ has the **finite model property**.

Definition (Finite Model Property)

A class of formulas Λ has the finite model property, if every satisfiable formula φ in the class Λ also has a finite model.

.

Finite Model Property

- Up to isomorphism, the finite structures of a given finite language are recursively enumerable.
- The property that a given finite structure is a model of a given FO-sentence is decidable.
- It follows that the satisfiability problem of every formula class with the finite model property is decidable.

Some Notions

- We fix a FO-language containing infinitely many predicate symbols of any arity, but no function nor constant symbols.
- Given a proposional modal logic **S**, we use **QS** to denote the corresponding FOML logic.
- A Kripke skeleton (W, R, D) is said to be **countably large** iff
  1. D is constant and countable.
  2. For some $w \in W$, the set of possible worlds $w^+ = \{v \mid Rwv\}$ is infinite.
- A **k-type** $t(x_1, \cdots, x_k)$ is a maximal consistent set of atomic and negated atomic formulas (including equalities). We often view a type as a quantifier-free formula that is the conjunction of its elements.

Decidability of Monadic Fragment of FOL

Lemma

Let ψ be a monadic FOL formula, possibly with equality. Suppose there are q variables and m predicates in ψ. If ψ is satisfiable, then it has a model of cardinality at most $q \cdot 2^m$.

Proof.

Let $\mathfrak{A} = (A, P_1^{\mathfrak{A}}, \cdots, P_m^{\mathfrak{A}}) \vDash \psi$. We define $f : A \to \{0,1\}^m$:

$$f(a) = \langle c_1, \cdots, c_m \rangle$$

where $c_i = 1$ iff $a \in P_i^{\mathfrak{A}}$.
For every $c \in \{0,1\}^m$, let $A_c = \{a \in A \mid f(a) = c\}$. Then we choose a set $B_c \subseteq A_c$ such that $B_c = A_c$ if $|A_c| \leq q$ and $|B_c| = q$ if $|A_c| > q$.

Decidability of Monadic Fragment of FOL

Lemma

Let ψ be a monadic FOL formula, possibly with equality. Suppose there are q variables and m predicates in ψ. If ψ is satisfiable, then it has a model of cardinality at most $q \cdot 2^m$.

Proof.

Let $\mathfrak{A} = (A, P_1^{\mathfrak{A}}, \cdots, P_m^{\mathfrak{A}}) \vDash \psi$. We define $f : A \to \{0,1\}^m$:

$$f(a) = \langle c_1, \cdots, c_m \rangle$$

where $c_i = 1$ iff $a \in P_i^{\mathfrak{A}}$.
For every $c \in \{0,1\}^m$, let $A_c = \{a \in A \mid f(a) = c\}$. Then we choose a set $B_c \subseteq A_c$ such that $B_c = A_c$ if $|A_c| \leq q$ and $|B_c| = q$ if $|A_c| > q$. Let $B = \bigcup_{c \in \{0,1\}^m} B_c$ and $\mathfrak{B} = \mathfrak{A}|_B$. Obviously, $|B| \leq q \cdot 2^m$ and $\mathfrak{B} \vDash \psi$ (by an EF-game $G_q(\mathfrak{A}, \mathfrak{B})$). □

Decidability of Monadic fragment of FOL

Corollary (Löwenheim, 1915)

The satisfiability problem for monadic FOL formulas is decidable.

Proof.

By the above lemma, the monadic fragment of FOL has finite model property. Therefore it's satisfiable problem is decidable. □

Undecidability of Monadic Fragment of FOML

Theorem (Kripke, 1962)

Let Σ be a set of monadic FOML sentences such that Σ contains all substitution instances of classically valid formulas of FO and **QS** *is valid on a countably large skeleton (W, R, D). If $\Sigma \subseteq$* **QS**, *then Σ is undecidable.*

Proof.

Reduce the decision problem for classically valid **dyadic** formulas to the decision problem for Σ. As we know, the validity of dyadic fragment is undecidable. For a dyadic formula ψ, let ψ^t be the formula obtained from ψ by replacing the atomic subformulas Sxy by $\Diamond(Px \wedge Qy)$.

Undecidability of Monadic Fragment of FOML

Theorem (Kripke, 1962)

Let Σ be a set of monadic FOML sentences such that Σ contains all substitution instances of classically valid formulas of FO and **QS** *is valid on a countably large skeleton (W, R, D). If $\Sigma \subseteq$* **QS**, *then Σ is undecidable.*

Proof.

Reduce the decision problem for classically valid **dyadic** formulas to the decision problem for Σ. As we know, the validity of dyadic fragment is undecidable. For a dyadic formula ψ, let ψ^t be the formula obtained from ψ by replacing the atomic subformulas Sxy by $\Diamond(Px \wedge Qy)$.

We show that ψ is valid iff $\psi^t \in \Sigma$.

Undecidability of Monadic Fragment of FOML

Theorem (Kripke, 1962)

Let Σ be a set of monadic FOML sentences such that Σ contains all substitution instances of classically valid formulas of FO and **QS** *is valid on a countably large skeleton (W, R, D). If $\Sigma \subseteq$* **QS***, then Σ is undecidable.*

Proof.

Reduce the decision problem for classically valid **dyadic** formulas to the decision problem for Σ. As we know, the validity of dyadic fragment is undecidable. For a dyadic formula ψ, let ψ^t be the formula obtained from ψ by replacing the atomic subformulas Sxy by $\Diamond(Px \wedge Qy)$.

We show that ψ is valid iff $\psi^t \in \Sigma$.

If ψ is valid, then $\psi^t \in \Sigma$ by definition. □

Undecidability of Monadic Fragment of FOML

cont'd

If ψ is not valid, then by Löwenheim-Skelom theorem, there is a countable FO structure $\mathfrak{A} = (A, I)$ such that $\mathfrak{A} \not\models \psi$.

Undecidability of Monadic Fragment of FOML

cont'd

If ψ is not valid, then by Löwenheim-Skelom theorem, there is a countable FO structure $\mathfrak{A} = (A, I)$ such that $\mathfrak{A} \nvDash \psi$.
We define a countable FOML model $\mathcal{M} = (W, R, D, \{V_w\}_{w \in W})$ where $D = A$. Let $w \in W$ be a point such that $w^+ = \{v \in W \mid Rwv\}$ is infinite and let $\rho : w^+ \rightarrow D$ be a surjection.

Undecidability of Monadic Fragment of FOML

cont'd

If ψ is not valid, then by Löwenheim-Skelom theorem, there is a countable FO structure $\mathfrak{A} = (A, I)$ such that $\mathfrak{A} \nvDash \psi$.
We define a countable FOML model $\mathcal{M} = (W, R, D, \{V_w\}_{w \in W})$ where $D = A$. Let $w \in W$ be a point such that $w^+ = \{v \in W \mid Rwv\}$ is infinite and let $\rho : w^+ \to D$ be a surjection.
If $v \notin w^+$, let $V_v(P) = V_v(Q) = \emptyset$. If $v \in w^+$, let $V_v(P) = \{\rho(v)\}$ and $V_v(Q) = \{b \in D \mid \langle \rho(v), b \rangle \in S^{\mathfrak{A}}\}$.
By induction it's not hard to show that for every dyadic formula $\psi(\overline{x})$, $\mathcal{M}, w \vDash \psi^t(\overline{a})$ iff $\mathfrak{A} \vDash \psi(\overline{a})$.

Decidability of FOL^2

Theorem (Scott 1963, Mortimer 1975)

The fragment with only two variables is decidable.

- Scott showed that the Sat problem for FO^2 can be reduced to the Sat problem for the $\forall\forall\exists^*$-class which is undecidable.
- Mortimer showed that FO^2 has the finite model property.
- Grädel, Kolaitis and Vardi(1997) gave a simpler proof and improved Mortimer's bound of model size.

Decidability of FO^2

Lemma

For any FO^2 sentence φ, there is a FO^2 sentence φ' such that

- *φ is satisfiable iff φ' is satisfiable.*
- *Every relation symbol occurring in φ' has arity at most 2.*
- *φ' has the form (**Scott Class**)*

$$\forall x \forall y\ \alpha(x,y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y\ \beta_i(x,y)$$

  where $\alpha(x,y)$ and $\beta_i(x,y)$ are quantifier-free formulas.
- *We may assume for every $i \leq m$, $\beta_i(x,y) \vDash x \neq y$.*

Decidability of FO^2

Definition

- A **k-type** $t(x_1, \cdots, x_k)$ is a maximal consistent set of atomic and negated atomic formulas (including equalities). We often view a type as a quantifier-free formula that is the conjunction of its elements.
- $\bar{a} = (a_1, \cdots, a_k)$ is a sequence of element of a structure $\mathfrak{A}$, then $t_{\bar{a}}$ is the unique k-type $t(z_1, \cdots, z_k)$ that $\bar{a}$ satisfies in $\mathfrak{A}$. If $t_{\bar{a}} = t$, we say that $\bar{a}$ **realizes** t.
- A element a of $\mathfrak{A}$ is a **king** if a is the only element that realizes the 1-type t_a on $\mathfrak{A}$, i.e. for all $b \neq a$, $t_b \neq t_a$.

Decidability of FO^2

- To construct a model of a FO^2 sentence θ, we need to first define its universe A and then specify the 1-types and 2-types realized by elements and pairs of elements from A.
- Since θ may contain equalities, a structure satisfying θ may have kings. But kings create obstructions in constructing models of a sentence. For example

$$\forall x \exists y (t(y) \wedge R(x,y)) \wedge \forall x \exists y (t(y) \wedge \neg R(x,y))$$

Decidability of FO^2

Theorem

Let θ be a sentence in the Scott class. If θ is satisfiable, then it has a finite model.

Proof.

Suppose that $\mathfrak{A} \models \theta$. Since $\mathfrak{A} \models \bigwedge_{i=1}^{m} \forall x \exists y \beta_i(x, y)$, there are functions $f_i : A \to A$ such that for every $a \in A$, $\mathfrak{A} \models \bigwedge_{i=1}^{m} \beta_i(a, f_i(a))$.

Decidability of FO^2

Theorem

Let θ be a sentence in the Scott class. If θ is satisfiable, then it has a finite model.

Proof.

Suppose that $\mathfrak{A} \models \theta$. Since $\mathfrak{A} \models \bigwedge_{i=1}^{m} \forall x \exists y \beta_i(x, y)$, there are functions $f_i : A \to A$ such that for every $a \in A$, $\mathfrak{A} \models \bigwedge_{i=1}^{m} \beta_i(a, f_i(a))$.
Let K be the set of all kings in $\mathfrak{A}$ and $C = K \cup \{g_i(k) \mid k \in K, 1 \leq i \leq m\}$ be the **court**.
$P = \{t_a \mid a \in \mathfrak{A}\}$ be the set of all 1-types realized in $\mathfrak{A}$, $Q \subseteq P$ be the set of 1-types realized by kings.
Let $n = |P - Q|$, We can enumerate all elements of $P - Q$ as $t_1, \cdots, t_n$.

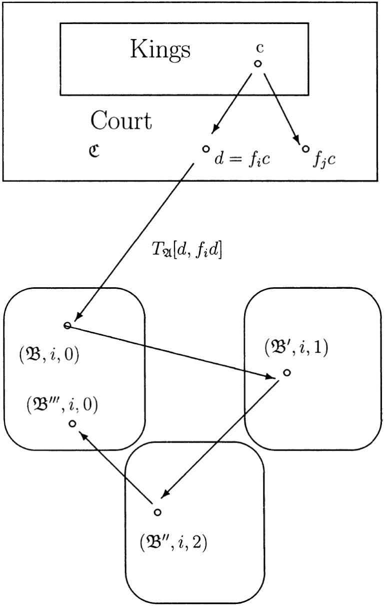
cont'd

The strategy of constructing a finite structure $\mathfrak{B}$ is

- Let D, E, F be disjoint sets of elements that are not in $\mathfrak{A}$. where $D = \{d_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ (resp. E, F).
- Let $B = C \cup D \cup E \cup F$ be the universe of $\mathfrak{B}$.
- $\mathfrak{B}$ has the same kings as $\mathfrak{A}$.
- To guarantee $\mathfrak{B} \models \forall x \forall y \alpha(x, y)$, we'll make sure every pair of B is assigned a 2-type realized by some pair of elements in $\mathfrak{A}$.
- We need to guarantee every element of B has **witness**, that is for every element $b \in B$ and every $i \leq m$ there is a $b_i \in B$ s.t. $\mathfrak{B} \models \beta(b, b_i)$.

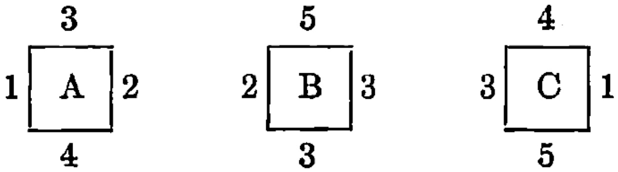
cont'd

- The kings will have witness in C by definition.
- For $b \in C - K$, if $f_i(b) \in C$, then b has a witness in C; if not, $t_{f_i(b)} \in P - Q$, let $t_{f_i(b)} = t_j$ and assign d_{ij} as the witness of b for $\beta_i(x, y)$. Moreover, we equip the pair (b, d_{ij}) with the 2-type of the pair $(b, g_i(b))$ on $\mathfrak{A}$.
- For $b \in D$, there is i, j s.t. $b = d_{ij}$. Thus b realizes 1-type t_j on $\mathfrak{B}$. Let t_a on $\mathfrak{A}$ is equal to t_j. Consider $g_i(a)$, if $g_i(a)$ is a king, then we assign $g_i(a)$ as witness of b and equip $(b, g_i(b)$ with 2-type $(a, g_i(a)$; if $g_i(a)$ is not a king, then $t_{g_i(a)}$ is equal to some type $t_l, l \leq n$. We assign e_{il} as witness, equip $(b, e_{il}$ with 2-type $(a, g_i(a))$ on $\mathfrak{A}$.
- $(D, E) \Rightarrow (E, F) \Rightarrow (F, D) \Rightarrow (D, E)$
- For every pair (b, b') for which 2-type has not been assigned, simply choose a pair of (a, a') of $\mathfrak{A}$ with t_a coincides with 1-type of b on $\mathfrak{B}$ (resp. a' and b').


Kings
c
Court
$\mathfrak{C}$
$d = f_i c$
$f_j c$
$T_{\mathfrak{A}}[d, f_i d]$
$(\mathfrak{B}, i, 0)$
$(\mathfrak{B}', i, 1)$
$(\mathfrak{B}''', i, 0)$
$(\mathfrak{B}'', i, 2)$

Tiling Problem (Wang, 1962)

A tile $\mathfrak{t}$ is a 1×1 square, each side of which has a color.



Given a finite set of tiles $\mathcal{T}$, can we cover up the whole plane with the same color on the common edge?

A	*B*	*C*
C	*A*	*B*
B	*C*	*A*

Undecidability of Tiling Problem

We can transform a Turing machine into a tile set.

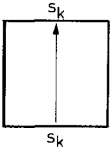


Fig. 12. Alphabet tile

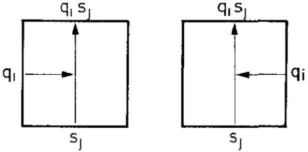


Fig. 13. Merging tiles

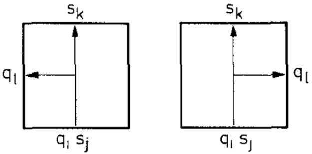


Fig. 14. Action tiles

Undecidability of Tiling Problem

Assume that the machine starts on a blank tape, then we can use following tiles in order to present its initial configuration.

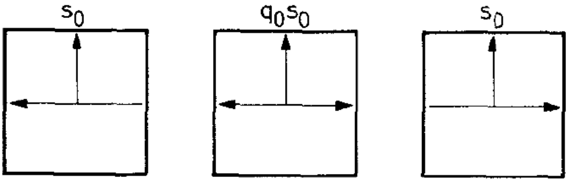


Fig. 15. Starting tiles for blank tape

Add a blank tile to the tiles set.

One can tile $\mathbf{Z} \times \mathbf{Z}$ iff the considered Turing machine does not halt.

Undecidability of $FOML^2$

Theorem (Kontchakov, Kurucz and Zakharyashev 2005)

Let **S** *be any propositional modal logic having a Kripke model that contains a point with infinitely many successors. Then the two-variable fragment of QS is undecidable.*

Proof.

Definition (Tiling Function)

$t = \langle u(t), d(t), r(t), l(t)\rangle$ is a tile. Let T is a set of tiles. A tiling function $\tau : \mathbf{N} \times \mathbf{N} \to T$ is a function satisfies that for all $i, j \in \mathbf{N}$,

$$u(\tau(i,j)) = d(\tau(i,j+1)) \quad \text{and} \quad r(\tau(i,j) = l(\tau(i+1,j))$$

□

cont'd

Given a finite set T, let χ_T be the $FOML^2$ sentence obtained as a conjunction of following formulas:

1. $\forall x \bigvee_{t \in T}(P_t(x) \wedge \bigwedge_{t' \neq t} \neg P_{t'}(x))$
2. $\forall x \forall y(H^+(x,y) \to \bigwedge_{r(t) \neq l(t')} \neg(P_t(x) \wedge P_{t'}(y)))$
3. $\forall x \forall y(V^+(x,y) \to \bigwedge_{u(t) \neq d(t')} \neg(P_t(x) \wedge P_{t'}(y)))$
4. $\forall x \exists y H^+(x,y) \wedge \forall x \exists y V^+(x,y)$
5. $\forall x \forall y(H^+(x,y) \to \Box H^+(x,y)) \wedge \forall x \forall y(V^+(x,y) \to \Box V^+(x,y))$
6. $\forall x \forall y(\Diamond V^+(x,y) \to V^+(x,y))$
7. $\forall x \Diamond Q(x)$
8. $\Box \forall x \forall y(V^+(x,y) \wedge \exists x(Q(x) \wedge H^+(y,x)) \to \forall y(H^+(x,y) \to \forall x(Q(x) \to V^+(y,x))))$

cont'd

We show that

$$\text{there is a } \tau : \mathbf{N} \times \mathbf{N} \to T \iff \chi_T \text{ is } \mathbf{QS}\text{-satisfiable}$$

cont'd

We show that

$$\text{there is a } \tau : \mathbf{N} \times \mathbf{N} \to T \iff \chi_T \text{ is } \mathbf{QS}\text{-satisfiable}$$

$\Rightarrow$: Suppose there is such a τ, let $w_0 \in W$ be a world s.t. w_0^+ is infinite. and let $\rho : w_0^+ \to \mathbf{N} \times \mathbf{N}$ be a surjective function. Identify D with $\mathbf{N} \times \mathbf{N}$ and define a model $\mathcal{M} = (W, R, D, \{V_w\}_{w \in W})$ as follows: for any $w \in W$,

- $V_w(Q) = \{\rho(v)\}$ if $v \in W_0^+$; $V_w(Q) = \emptyset$ if $v \notin W_0^+$;
- $V_w(P_t) = \{(i,j) \mid \tau(i,j) = t\}$
- $V_w(H^+) = \{\langle (i_1, j), (i_2, j) \rangle \mid i_2 = i_1 + 1\}$
- $V_w(V^+) = \{\langle (i, j_1), (i, j_2) \rangle \mid j_2 = j_1 + 1\}$

cont'd

We show that

$$\text{there is a } \tau : \mathbf{N} \times \mathbf{N} \to T \iff \chi_T \text{ is } \mathbf{QS}\text{-satisfiable}$$

$\Rightarrow$: Suppose there is such a τ, let $w_0 \in W$ be a world s.t. w_0^+ is infinite. and let $\rho : w_0^+ \to \mathbf{N} \times \mathbf{N}$ be a surjective function. Identify D with $\mathbf{N} \times \mathbf{N}$ and define a model $\mathcal{M} = (W, R, D, \{V_w\}_{w \in W})$ as follows: for any $w \in W$,

- $V_w(Q) = \{\rho(v)\}$ if $v \in W_0^+$; $V_w(Q) = \emptyset$ if $v \notin W_0^+$;
- $V_w(P_t) = \{(i, j) \mid \tau(i, j) = t\}$
- $V_w(H^+) = \{\langle (i_1, j), (i_2, j) \rangle \mid i_2 = i_1 + 1\}$
- $V_w(V^+) = \{\langle (i, j_1), (i, j_2) \rangle \mid j_2 = j_1 + 1\}$

It's easy to check $\mathcal{M}, w_0 \models \chi_T$.

cont'd

$\Leftarrow$: Suppose that χ_T is **QS**-satisfiable, i.e. there is a $\mathcal{M}$ and $v \in W_M$ s.t $\mathcal{M}, v \vDash \chi_T$. The conjunction of (4)-(8) imply in **QS** the formula

$$(9) \quad \forall x \forall y \forall z (H^+(x,y) \wedge V^+(x,z) \rightarrow \exists x (H^+(z,x) \wedge V^+(y,x)))$$

Thus $\mathcal{M}, v \vDash (9)$. (9) and (4) imply that for every $i, j \in \mathbf{N}$ there are elements $a_{ij} \in D_v$ s.t $\mathcal{M}, v \vDash H^+(a_{ij}, a_{i+1,j})$ and $\mathcal{M}, v \vDash V^+(a_{ij}, a_{i,j+1})$. Since (1)-(3) hold in v, it's easily seen that the function defined by

$$\tau(i,j) = t \quad \text{iff} \quad \mathcal{M}, v \vDash P_t(a_{ij})$$

tiles $\mathbf{N} \times \mathbf{N}$.

Conclusion

- We compare the decision problem of the monadic fragment and the two-variable fragment of FOL and FOML.
- We find that the undecidable fragments are much more common in FOML than FOL and ML.
- The monadic and the two-variable fragments of practically all FOMLs is undecidable.
- We can find some FOML fragments which are decidable e.g. only one-variable FOML fragment but its expressivity is very limited.
- Monodic fragment.

References I

Börger, E., Grädel, E., and Gurevich, Y. (2001).
The classical decision problem.
Springer Science & Business Media.

Brauner, T. and Ghilardi, S. (2007).
First Order Modal Logic.
Kluwer Academic,.

Church, A. (1936).
A note on the entscheidungsproblem.
Journal of Symbolic Logic, 1(1):40–41.

Frank, W. and Michael, Z. (2001).
Decidable fragments of first-order modal logics.
Journal of Symbolic Logic, 66(3):1415–1438.

References II

Grädel, E., Kolaitis, P. G., and Vardi, M. Y. (1997).
On the decision problem for two-variable first-order logic.
Bulletin of Symbolic Logic, 3(1):53–69.

Kontchakov, R., Kurucz, A., and Zakharyaschev, M. (2005).
Undecidability of first-order intuitionistic and modal logics with two variables.
Bulletin of Symbolic Logic, 11(3):428–438.

Kripke, S. A. (1962).
The undecidability of monadic modal quantification theory.
Z. Math. Logik Grundlagen Math., 8:113–116.

Löwenheim, L. (1915).
Über möglichkeiten im relativkalkl.
Mathematische Annalen, 76(4):447–470.

References III

Mortimer, M. (1975).
On languages with two variables.
Mathematical Logic Quarterly, 21(1):135–140.

Robinson, R. M. (1971).
Undecidability and nonperiodicity for tilings of the plane.
Inventiones Mathematicae, 12(3):177–209.

Scott, D. (1962).
A decision method for validity of sentences in two variables.
Journal of Symbolic Logic - JSYML, 27:477.

References IV

Turing, A. M. (1937).
Turing, a.m.: On computable numbers, with an application to the entscheidungs problem.
Proceedings of the London Mathematical Society, 42(1):230–265.

Wang, H. (1963).
Dominoes and the AEA case of the decision problem.
In *Proc. Sympos. Math. Theory of Automata (New York, 1962)*, pages 23–55. Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.